Kuwait University	Math 101	Date:	December 14, 2006
Dept. of Math. & Comp. Sci.	Second Exam	Duration:	90 minutes

Calculators, mobile phones, pagers and all other mobile communication equipment are not allowed

Answer the following questions:

1. Use differentials to approximate $1 - \frac{1}{\sqrt[3]{8.01}}$. (4 pts.)

2. Find an equation of the tangent line to the curve of

$$2y + y^2 \tan x + \sin(x^2 y) - 2 = 0, \text{ at } x = 0.$$ (4 pts.)

3. As a right circular cylinder is being heated, its radius is increasing at a rate of 0.04 mm/sec and its height is increasing at a rate of 0.15 mm/sec. Find the rate at which the volume of the cylinder is changing when the radius is 0.5 mm and the height is 0.3 mm. (4 pts.)

4. (a) State The Mean Value Theorem. (1 pt.)

   (b) Let $f'(x) = \frac{1}{3 + 2x^2}$, for all real number x and $f(1) = 0$. Show that

$$\frac{1}{11} < f(2) < \frac{1}{5}.$$ (4 pts.)

5. Let $f(x) = \frac{x}{x^2 - 1}$ and given that $f'(x) = -\frac{x^2 + 1}{(x^2 - 1)^2}$ and $f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}$.

   (a) Find the vertical and horizontal asymptotes for the graph of f, if any.

   (b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

   (c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

   (d) Sketch the graph of f. (8 pts.)

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1. Take $x_\circ = 8, \Delta x = 0.01$ & $f(x) = 1 - \dfrac{1}{\sqrt[3]{x}} \implies f'(x) = \frac{1}{3x^{\frac{4}{3}}}, f(8) = \dfrac{1}{2}, f'(8) = \frac{1}{48}$

   Thus: $1 - \dfrac{1}{\sqrt[3]{8.01}} \simeq f(8) + f'(8)(0.01) = \dfrac{1}{2} + \frac{1}{48}(0.01) = \boxed{0.500\,21}$

2. At $x = 0$, $y = 1$. Differentiate w.r.t $x \implies 2y + y^2 \tan x + \sin(x^2 y) - 2 = 0$

   $2y' + 2yy' \tan x + y^2 \sec^2 x + (2xy + x^2 y') \cos(x^2 y) = 0 \implies y'|_{(0,1)} = -\frac{1}{2}$.

   Equation of tangent line: $\boxed{y - 1 = -\frac{1}{2}x}$ or $\boxed{x + 2y - 2 = 0}$.

3. $V = \pi r^2 h$.

   Differentiate w.r.t. t, $\dfrac{dV}{dt} = \pi\left(2rh\dfrac{dr}{dt} + r^2\dfrac{dh}{dt}\right) \implies \left.\dfrac{dV}{dt}\right|_{\substack{r=0.5\\h=0.3}} = 0.049\,5\ \pi\ \text{mm}^3/\text{sec}.$

4. (b) $f'(x) = \dfrac{1}{3 + 2x^2}$ exists for all x $[3 + 2x^2 \neq 0]$. Then f is continuous for all x.

   Thus, f is continuous on $[1, 2]$ and differentiable in $(1, 2)$. From The Mean Value Theorem, $\exists\ c \in (1, 2)$ such that

   $f(2) - f(1) = f'(c)(2 - 1) \implies f(2) - 0 = \frac{1}{3+2c^2}(2 - 1) \implies \boxed{f(2) = \frac{1}{3+2c^2}}$.

   As $1 < c < 2 \implies 1 < c^2 < 4 \implies 5 < 3 + 2c^2 < 11 \implies \frac{1}{5} > \frac{1}{3+2c^2} > \frac{1}{11}$.

5. $D_f = \mathbb{R} - \{-1, 1\}$, $f(0) = 0$. $f(-x) = -f(x) \implies f$ is odd function and the graph of f is symmetric about the origin.

   (a) $\lim_{x\to 1^\pm} f(x) = \pm\infty \implies \boxed{x = 1 \text{ is V. A}}$.& f has infinite discontinuity at $x = 1$.

   $\lim_{x\to -1^\pm} f(x) = \mp\infty \implies \boxed{x = -1 \text{ is V. A}}$.& f has infinite discontinuity at $x = -1$.

   $\lim_{x\to \pm\infty} f(x) = 0 \implies \boxed{y = 0 \text{ is H. A.}}$

   $f(x) = 0 \implies x = 0$.The graph of f intersects the H.A ($y = 0$) at $x = 0$.

   (b) $f'(x) \neq 0$ & (f has infinite discontinuity at $\boxed{x = \pm 1}$, where $f'(x)$ does not exist.).

I	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
sign of $f'(x)$	$-$	$-$	$-$
Conclusion	↘	↘	↘

   f is decreasing on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. f has no local extremum.

   (c) $f''(x) = 0 \implies \boxed{x = 0}$.($f$ is not continuous at $\boxed{x = \pm 1}$, where $f''(x)$ does not exist).

I	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
sign of $f''(x)$	$-$	$+$	$-$	$+$
Concavity	CD	CU	CD	CU

   $\implies$ $(0, 0)$ is a point of inflection.

   (d)